Availability Analysis of the k-out-of-n:G system using Markov Model and Supplementary Variable Technique

M. A. El-Damcese, N. H. El-Sodany

Abstract — Certain stochastic processes with discrete states in continuous time can be converted into Markov process by the well-known method of including supplementary variables technique. This paper presents Markov models for analyzing the availability for the k-out-of-n:G system with three types of failure (partial failure, complete failure under repair and complete failure under maintenance) using supplementary variable technique. The Markov method is used to develop generalized expressions for system state probabilities and system's availability. An illustrative example is presented in order to illustrate the performance of the model. The transient and steady states have been presented when the failure and repair rates are variables and constants respectively. Lagrange's method for partial differential equations is used to solve system governing equations when the failure and repair rates are variables. When both the failure and repair rates are constants, the system of differential equations with the initial conditions has been solved using Laplace transformation by the aid of Maple program. The steady state availability when both failure and repair rates are constants, has been computed.

Keywords — k-out-of-n:G system, supplementary variable technique, Markov model, partial failure, availability analysis, Laplace transformation, Lagrange's method.

1 INTRODUCTION

The study of repairable systems is a basic and important topic in reliability engineering. The system reliability and the system availability play an increasingly important role in industrial systems, and manufacturing systems. In order to find the availability of a system one has to form a system of linear differential equations using mnemonic rule. This rule states that the derivative of the probability of every state is equal to the sum of all probability flows which come from other states to the given state minus the sum of all probability flows which go out from the given state to the other states. The differential equations thus derived are known as the Chapman-Kolmogorov differential equations.

Throughout the history of reliability theory, large numbers of reliability problems were solved by using reliability models. There are several methods to establish such models. Among them the supplementary variable technique plays an important role. In case the failure and repair rates are variables then the sytem loses its Markovian property. By Introducing supplementary variables, the non-Marlovian character of the system is changed to Markovian. Several authors have studied the reliability of the various systems using supplementary variable technique. [4] first put forward the supplementary variable technique and established the M/G/1 queeuing model. After that, the supplementary variable technique was used by many authors to solve a good number of queuing problems.

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[10] analyzed N/G/1 finite queue with the supplementary variable method. In the steady-state case, many problems are more readily treated by the supplementary variable technique than by the imbedded Markov chain. [8] first used the supplementary variable technique to study a reliability model. After that, other researchers widely applied this idea to study many reliability problems. [1] discussed the supplementary variable technique in stochastic model. [12] analyzed the reliability of polytube tube industry using supplementary variable Technique. [13] used supplementary variable technique for problem formulation. [15] studied the stochastic behavior of standby system under preemptive priority repair and obtained the expression for transient and steady state of the system using techniques of supplementary variables and Laplace transforms.

The problem of evaluating the availability and reliability of the k-out-of-n:G system has been subject of many studies throughout literature. Several authors have considered k-out-of-n:G systems. [6] considered circular consecutive k-out-of-n:G systems. He has applied continuous-time homogeneous Markov process to evaluate availability, reliability and MTTF for circular consecutive-k-out-of-n:G system with repairman, and when the system displays a gradual degradation of its performance, its availability and reliability are analyzed in terms of fuzzy success states. [5] analyzed the k-out-of-n:G system with human errors, common cause and time dependent system. [11] considered k-out-of-n:G systems with M failure modes. He obtained closed form solutions of the transient probabilities, reliability, and the mean time to failure (MTTF) for non-repairable systems. For repairable systems he suggested numerical solutions to obtain the reliability and the mean time between failures (MTBF). [7] studied the dynamic behavior of the k-out-of-n:G systems. [14] studied unavailability analysis for k-out-of-n:G systems with multiple

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failure modes based on micro-Markov models. [3] found an exact equation and an algorithm for reliability evaluation of k-out-of-n:G system. [2] studied a direct method for reliability computation of k-out-of-n:G systems.

This paper presents Markov method and supplementary variable technique to obtain a general formula for the availability of the k-out-of-n:G system of n-identical and independent components subject to three types of failures and the repaired component is good as new. A component can fail either due to a partial failure or complete failure under repair or due to complete failure under maintenance. In this paper we consider that the system and its components has three states: up, degraded, and down. The transition from up state to degraded state represents a partial failure, and the transition from up state to down (failed) state or from degraded state to down (failed) state represents a complete failure.

The outline of the paper is as follows. The basic assumptions and notations used are given in Section 2. Section 3 is voted to the analysis of k-out-of-n:G system. The availability of the system is obtained in this section using state probabilities, initial and boundary conditions. In Section 4 we give a mathematical modeling of the system when $n=3$ and $k=2$ and the availability of the system is obtained when the failure and repair rates are variables using Lagrange's method, and when the failure and repair rates are constants using Laplace transformation and also we obtained the steady state availability when the failure and repair rates are constants. In Section 5 we obtained the numerical solutions of the 2-out-of-3:G system when both failure and repair rates are constants by the aid of Maple program. Some concluding remarks are given in Section 6.

2 MODEL DESCRIPTION

We develop the Markov model and supplementary variable technique for n-component system and these components are identical and repairable. At time $t=0$ the system is considered to be in good state and it fails when at least k of the n components fail. The system or the components may fail either due to partial failure or complete failure under repair or complete failure under maintenance. The assumptions and notations, on which the present analysis is based upon, are as follows:

- The system is composed of n-identical and independent components.
- At time $t=0$ all components are up, and the system can work if and only if at least k of the n components work (or are good).
- All Failure and repair rates of the components are taken as variables. So, this process is non-Markovian and we will use the supplementary variable technique to convert it into Markovian process. By using supplementary variable technique, we can construct the differential equations associated with the model.
- The repaired unit is as good as new one for a specified duration.
- Each component is failed with failure rate $\lambda_i(y), i=1,2,3,4$ after an elapsed failure time y.
- The failed component is repaired with repair rate $\mu_i(x), i=1,2,3$ after an elapsed repair time x.
- Failure and repair rates, $\lambda_i(y)$ and $\mu_i(x)$, are independent of each other.
- The system cannot return to the working condition when $n-k+1$ components completely fail.

Notations:

n	:	the number of components in the system.
k	:	the minimum number of components that must work for the k-out-of-n:G system to work.
(m_1,m_2,m_3)	:	the state of the system, where m_1, m_2 and m_3 represent the number of failed components due to partial failure, complete failure under repair and complete failure under maintenance, respectively.
$\lambda_1(y)$	:	the failure rate of a component when it goes from up state to degraded state after an elapsed failure time y.
$\lambda_2(y)$	:	the failure rate of a component when it goes from up state to down state under repair after an elapsed failure time y.
$\lambda_3(y)$	:	the failure rate of a component when it goes from up state to down state under maintenance after an elapsed failure time y.
$\lambda_4(y)$	:	the failure rate of a component when it goes from degraded state to down state under repair after an elapsed failure time y
$\mu_1(x)$	:	the repair rate of a component when it goes from degraded state to up state after an elapsed repair time x.
$\mu_2(x)$	:	the repair rate of a component when it goes from down state under repair to up state after an elapsed repair time x.
$\mu_3(x)$	:	the repair rate of a component when it goes from down state under maintenance to up state after an elapsed repair time x.
$P_i(x,y,t)$	:	probability that the system is in state i at time t and has an elapsed failure time y and elapsed repair time x, where i is the state (m_1,m_2,m_3), $m_1,m_2,m_3=0,1,\dots,n$.
$P_i(s)$	:	Laplace transformation of $P_i(t)$.

3 ANALYSIS OF THE K-OUT-OF-N:G SYSTEM

State probabilities:

The state probabilities of the system $P_i(x,y,t)$, where i represent the state (m_1,m_2,m_3), $m_1,m_2,m_3=0,1,\dots,n$, can be viewed as a result of solving the following set of first order differential equations by assuming that:

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$i=(m_1,m_2,m_3),\ a=(m_1+1,m_2,m_3),\ b=(m_1,m_2+1,m_3)$

$,\ c=(m_1,m_2,m_3+1),\ d=(m_1-1,m_2,m_3),\ h=(m_1,m_2-1,m_3)$

$,\ f=(m_1,m_2,m_3-1),\ g=(m_1+1,m_2-1,m_3)$

For $m_1=m_2=m_3=0$:

$$\left(\frac{d}{dt}+n\sum_{i=1}^{3}\lambda_i(y)\right)P_i(t)=\int\mu_1(x)P_a(x,y,t)\,dx\,dy \\ +\int\mu_2(x)P_b(x,y,t)\,dx\,dy+\int\mu_3(x)P_c(x,y,t)\,dx\,dy \tag{1.1}$$

For $1\le m_1\le n,\ m_2=m_3=0$:

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+(n-m_1)\sum_{i=1}^{3}\lambda_i(y)+m_1\lambda_4(y)+m_1\mu_1(x)\right) \\ P_i(\mathrm{x,y,t})=(n-m_1+1)\lambda_1(y)P_d(\mathrm{x,y,t}) \\ +(m_1+1)\mu_1(x)P_a(\mathrm{x,y,t})+\mu_2(x)P_b(\mathrm{x,y,t}) \\ +\mu_3(x)P_c(\mathrm{x,y,t}) \tag{1.2}$$

For $0\le m_1<n,\quad 0\le m_2,m_3\le n-k$:

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\left(n-\sum_{i=1}^{3}m_i\right)\sum_{i=1}^{3}\lambda_i(y)+m_1\lambda_4(y)+\sum_{i=1}^{3}m_i\mu_i(x)\right) \\ P_i(\mathrm{x,y,t})=(m_1+1)\mu_1(x)P_a(\mathrm{x,y,t})+(m_2+1)\mu_2(x)P_b(\mathrm{x,y,t}) \\ +(m_3+1)\mu_3(x)P_c(\mathrm{x,y,t})+\left(n-\sum_{I=1}^{3}m_i+1\right) \\ \left(\lambda_1(y)P_d(\mathrm{x,y,t})+\lambda_2(y)P_h(\mathrm{x,y,t})+\lambda_3(y)P_f(\mathrm{x,y,t})\right) \\ +(m_1+1)\lambda_4(y)P_g(\mathrm{x,y,t}) \tag{1.3}$$

For $m_2+m_3=n-k+1,\quad m_1=0$:

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\sum_{i=1}^{3}m_i\mu_i(x)\right)P_i(\mathrm{x,y,t})=\left(n-\sum_{I=1}^{3}m_i+1\right) \\ \left(\lambda_1(y)P_d(\mathrm{x,y,t})+\lambda_2(y)P_h(\mathrm{x,y,t})+\lambda_3(y)P_f(\mathrm{x,y,t})\right) \\ +(m_1+1)\lambda_4(y)P_g(\mathrm{x,y,t}) \tag{1.4}$$

For $m_2+m_3=n-k+1,\quad m_1=1,\dots,k-1$:

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\sum_{i=2}^{3}m_i\mu_i(x)\right)P_i(\mathrm{x,y,t})=\left(n-\sum_{I=1}^{3}m_i+1\right) \\ \left(\lambda_1(y)P_d(\mathrm{x,y,t})+\lambda_2(y)P_h(\mathrm{x,y,t})+\lambda_3(y)P_f(\mathrm{x,y,t})\right) \\ +(m_1+1)\lambda_4(y)P_g(\mathrm{x,y,t}) \tag{1.5}$$

Initial Conditions:

As elapsed failure and repair time are zero initially and the system is completely in working state, the initial conditions thus becomes:

$$P_i(0)=1 \text{ for } \mathrm{i}=(m_1,m_2,m_3)\ ,\ m_1=0,\ m_2=0,\ m_3=0 \\ P_i(x,y,0)=0 \text{ for } \mathrm{i}=(m_1,m_2,m_3)\ ,\ 1\le m_1+m_2+m_3\le n \tag{2}$$

Boundary conditions:

Since a system is in the failed state at time t with failure rate $\lambda_i(y)$ but repair has not been done at that time, so the boundary conditions are specified as:

For $1\le m_1\le n,\ m_2=m_3=0$:

$$P_i(0,\mathrm{y,t})=\int(n-m_1+1)\lambda_1(y)P_d(\mathrm{x,y,t})\,dx \tag{3.1}$$

For $0\le m_1<n,\quad 0\le m_2,m_3\le n-k$:

$$P_i(0,\mathrm{y,t})=\left(n-\sum_{I=1}^{3}m_i+1\right) \\ \int\left(\lambda_1(y)P_d(\mathrm{x,y,t})+\lambda_2(y)P_h(\mathrm{x,y,t})+\lambda_3(y)P_f(\mathrm{x,y,t})\right)dx \\ +\int(m_1+1)\lambda_4(y)P_g(\mathrm{x,y,t})\,dx \tag{3.2}$$

For $m_2+m_3=n-k+1,\quad m_1=0$:

$$P_i(0,\mathrm{y,t})=\left(n-\sum_{I=1}^{3}m_i+1\right) \\ \int\left(\lambda_1(y)P_d(\mathrm{x,y,t})+\lambda_2(y)P_h(\mathrm{x,y,t})+\lambda_3(y)P_f(\mathrm{x,y,t})\right)dx \\ +\int(m_1+1)\lambda_4(y)P_g(\mathrm{x,y,t})\,dx \tag{3.3}$$

For $m_2+m_3=n-k+1,\quad m_1=1,\dots,k-1$:

$$P_i(0,\mathrm{y,t})=\left(n-\sum_{I=1}^{3}m_i+1\right) \\ \int\left(\lambda_1(y)P_d(\mathrm{x,y,t})+\lambda_2(y)P_h(\mathrm{x,y,t})+\lambda_3(y)P_f(\mathrm{x,y,t})\right)dx \\ +\int(m_1+1)\lambda_4(y)P_g(\mathrm{x,y,t})\,dx \tag{3.4}$$

The system of differential Eqs.(1.1-1.5) together with the boundary conditions Eqs.(3.1-3.4) and initial conditions Eq.(2) are called Chapman- Kolmogorov differential difference equations. Eq.(1.1) is a linear differential equation of first order and other Eqs.(1.2-1.5) are linear partial differential equations. In order to find the availability of the system, the governing Eqs.(1.1-1.5) will be solved along with the initial and boundary conditions.

The system availability is the summation of the probabilities of all working states, the general form solution of the k-out-of-n:G system availability at time t is given by:

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$$A(t)=\sum_i P_i(x,y,t) \tag{4}$$

$$, i=(m_1,m_2,m_3) \quad , m_2+m_3 \neq n-k+1$$

Here, according to the values of k and n we use the numerical method based on the Runge-Kutte 4th order method to find the solution of the system availability $A(t)$ of Eq.(4) with the initial conditions Eq.(2) and boundary conditions Eqs.(3.1-3.4).

4 MATHEMATICAL MODELING OF THE K-OUT-OF-N:G SYSTEM

Consider the k-out-of-n:G system when $k=2$ and $n=3$. Table 1 gives the event space of states of the system.

Table 1: Event space of states of the system

m	(m_1,m_2,m_3)
0	$(0,0,0)$
1	$(1,0,0)$, $(0,1,0)$, $(0,0,1)$
2	$(1,1,0)$, $(0,1,1)$, $(1,0,1)$, $(2,0,0)$, $(0,2,0)$, $(0,0,2)$
3	$(1,1,1)$, $(1,2,0)$, $(1,0,2)$, $(2,1,0)$, $(2,0,1)$, $(3,0,0)$

Based on the above notations and assumptions, the state transition diagram of the 2-out-of-3:G system is given in Fig. 1.

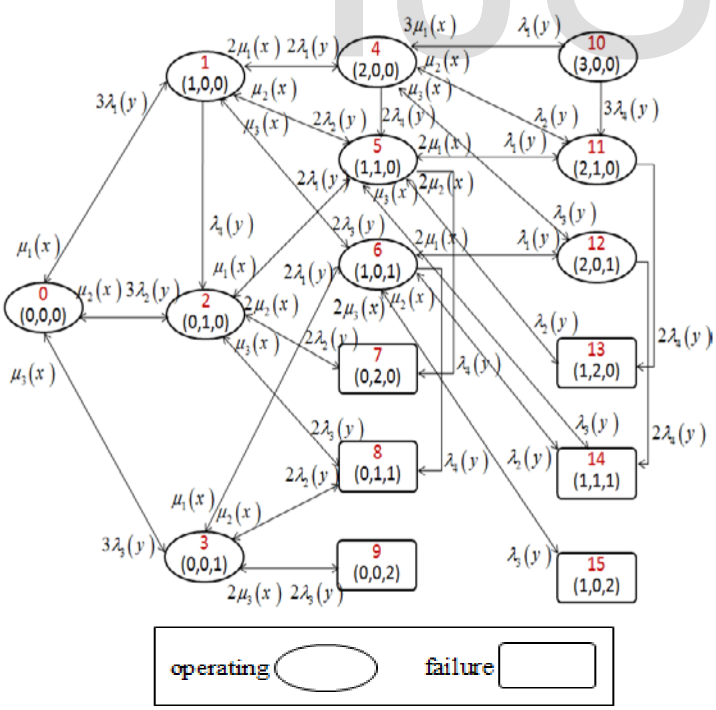

Fig. 1: State transition diagram of the 2-out-of-3: G system

Probability considerations gives the following system of differential difference equations associated with the state transition diagram (Fig. 1). We first develop the differential difference equations in transient state by using mnemonic rule, as under:

4.1 Transient State When Both Failure And Repair Rates Are Variables

The set of differential equations associated with the 2-out-of-3:G system are given by:

$$\left(\frac{d}{dt}+3\sum_{i=1}^{3}\lambda_i(y)\right)P_0(t)=C_0 \tag{5.1}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+T_1(x,y)\right)P_1(x,y,t)=C_1(x,y,t) \tag{5.2}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+T_2(x,y)\right)P_2(x,y,t)=C_2(x,y,t) \tag{5.3}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+T_3(x,y)\right)P_3(x,y,t)=C_3(x,y,t) \tag{5.4}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+T_4(x,y)\right)P_4(x,y,t)=C_4(x,y,t) \tag{5.5}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+T_5(x,y)\right)P_5(x,y,t)=C_5(x,y,t) \tag{5.6}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+T_6(x,y)\right)P_6(x,y,t)=C_6(x,y,t) \tag{5.7}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+2\mu_2(x)\right)P_7(x,y,t)=C_7(x,y,t) \tag{5.8}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+T_8(x,y)\right)P_8(x,y,t)=C_8(x,y,t) \tag{5.9}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+2\mu_3(x)\right)P_9(x,y,t)=2\lambda_3(y)P_3(x,y,t) \tag{5.10}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+T_{10}(x,y)\right)P_{10}(x,y,t)=\lambda_1(y)P_4(x,y,t) \tag{5.11}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+T_{11}(x,y)\right)P_{11}(x,y,t)=C_{11}(x,y,t) \tag{5.12}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+T_{12}(x,y)\right)P_{12}(x,y,t)=C_{12}(x,y,t) \tag{5.13}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+2\mu_2(x)\right)P_{13}(x,y,t)=C_{13}(x,y,t) \tag{5.14}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+T_{14}(x,y)\right)P_{14}(x,y,t)=C_{14}(x,y,t) \tag{5.15}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+2\mu_3(x)\right)P_{15}(x,y,t)=\lambda_3(y)P_6(x,y,t) \tag{5.16}$$

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where,

$$C_0(t)=\int\mu_1(x)P_1(x,y,t)dx\ dy+\int\mu_2(x)P_2(x,y,t)dx\ dy$$
$$+\int\mu_3(x)P_3(x,y,t)dx\ dy$$

$$T_1(x,y)=2\sum_{i=1}^{3}\lambda_i(y)+\lambda_4(y)+\mu_1(x)$$

$$C_1(x,y,t)=2\mu_1(x)P_4(x,y,t)+\mu_2(x)P_5(x,y,t)$$
$$+\mu_3(x)P_6(x,y,t)+3\lambda_1(y)P_0(t)$$

$$T_2(x,y)=2\sum_{i=1}^{3}\lambda_i(y)+\mu_2(x)$$

$$C_2(x,y,t)=\mu_1(x)P_5(x,y,t)+2\mu_2(x)P_7(x,y,t)$$
$$+\mu_3(x)P_8(x,y,t)+3\lambda_2(y)P_0(t)+\lambda_4(y)P_1(x,y,t)$$

$$T_3(x,y)=2\sum_{i=1}^{3}\lambda_i(y)+\mu_3(x)$$

$$C_3(x,y,t)=\mu_1(x)P_6(x,y,t)+\mu_2(x)P_8(x,y,t)$$
$$+2\mu_3(x)P_9(x,y,t)+3\lambda_3(y)P_0(t)$$

$$T_4(x,y)=\sum_{i=1}^{3}\lambda_i(y)+2\lambda_4(y)+2\mu_1(x)$$

$$C_4(x,y,t)=3\mu_1(x)P_{10}(x,y,t)+\mu_2(x)P_{11}(x,y,t)$$
$$+\mu_3(x)P_{12}(x,y,t)+2\lambda_1(y)P_1(x,y,t)$$

$$T_5(x,y)=\sum_{i=1}^{3}\lambda_i(y)+\lambda_4(y)+\mu_1(x)+\mu_2(x)$$

$$C_5(x,y,t)=2\mu_1(x)P_{11}(x,y,t)+2\mu_2(x)P_{13}(x,y,t)$$
$$+\mu_3(x)P_{14}(x,y,t)+2\lambda_2(y)P_1(x,y,t)$$
$$+2\lambda_4(y)P_4(x,y,t)+2\lambda_1(y)P_2(x,y,t)$$

$$T_6(x,y)=\sum_{i=1}^{3}\lambda_i(y)+\lambda_4(y)+\mu_1(x)+\mu_3(x)$$

$$C_6(x,y,t)=2\mu_1(x)P_{12}(x,y,t)+\mu_2(x)P_{14}(x,y,t)$$
$$+2\mu_3(x)P_{15}(x,y,t)+2\lambda_1(y)P_3(x,y,t)$$
$$+2\lambda_3(y)P_1(x,y,t)$$

$$C_7(x,y,t)=2\lambda_2(y)P_2(x,y,t)+\lambda_4(y)P_5(x,y,t)$$

$$T_8(x,y)=\mu_2(x)+\mu_3(x)$$

$$C_8(x,y,t)=2\lambda_2(y)P_3(x,y,t)+2\lambda_3(y)P_2(x,y,t)$$
$$+\lambda_4(y)P_6(x,y,t)$$

$$T_{10}(x,y)=3\mu_1(x)+3\lambda_4(y)$$

$$T_{11}(x,y)=2\mu_1(x)+\mu_2(x)+2\lambda_4(y)$$

$$C_{11}(x,y,t)=\lambda_1(y)P_5(x,y,t)+\lambda_2(y)P_4(x,y,t)$$
$$+3\lambda_4(y)P_{10}(x,y,t)$$

$$T_{12}(x,y)=2\mu_1(x)+\mu_3(x)+2\lambda_4(y)$$

$$C_{12}(x,y,t)=\lambda_1(y)P_6(x,y,t)+\lambda_3(y)P_4(x,y,t)$$

$$C_{13}(x,y,t)=\lambda_2(y)P_5(x,y,t)+2\lambda_4(y)P_{11}(x,y,t)$$

$$T_{14}(x,y)=\mu_2(x)+\mu_3(x)$$

$$C_{14}(x,y,t)=\lambda_2(y)P_6(x,y,t)+\lambda_3(y)P_5(x,y,t)$$
$$+2\lambda_4(y)P_{12}(x,y,t)$$

Initial Conditions:

at time $t=0$, the state probabilities given by Eqs.(5.1-5.16) satisfy the following initial conditions:

$$P_0(0)=1$$
$$P_i(x,y,0)=0 \qquad ,i=1,2,....,15 \tag{6}$$

Boundary Conditions:

the boundary conditions of the system are specified as:

$$P_1(0,y,t)=3\lambda_1(y)P_0(t) \tag{7.1}$$

$$P_2(0,y,t)=3\lambda_2(y)P_0(t)+\int\lambda_4(y)P_1(x,y,t)dx \tag{7.2}$$

$$P_3(0,y,t)=3\lambda_3(y)P_0(t) \tag{7.3}$$

$$P_4(0,y,t)=\int 2\lambda_1(y)P_1(x,y,t)dx \tag{7.4}$$

$$P_5(0,y,t)=\int 2\lambda_2(y)P_1(x,y,t)dx+\int 2\lambda_4(y)P_4(x,y,t)dx$$
$$+\int 2\lambda_1(y)P_2(x,y,t)dx \tag{7.5}$$

$$P_6(0,y,t)=\int 2\lambda_1(y)P_3(x,y,t)dx$$
$$+\int 2\lambda_3(y)P_1(x,y,t)dx \tag{7.6}$$

$$P_7(0,y,t)=\int 2\lambda_2(y)P_2(x,y,t)dx+\int\lambda_4(y)P_5(x,y,t)dx \tag{7.7}$$

$$P_8(0,y,t)=\int 2\lambda_2(y)P_3(x,y,t)dx+\int 2\lambda_3(y)P_2(x,y,t)dx$$
$$+\int\lambda_4(y)P_6(x,y,t)dx \tag{7.8}$$

$$P_9(0,y,t)=\int 2\lambda_3(y)P_3(x,y,t)dx \tag{7.9}$$

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$$P_{10}(0,y,t)=\int\lambda_1(y)P_4(x,y,t)dx \quad (7.10)$$

$$P_{11}(0,y,t)=\int\lambda_1(y)P_5(x,y,t)dx+\int\lambda_2(y)P_4(x,y,t)dx+\int3\lambda_4(y)P_{10}(x,y,t)dx \quad (7.11)$$

$$P_{12}(0,y,t)=\int\lambda_1(y)P_6(x,y,t)dx+\int\lambda_3(y)P_4(x,y,t)dx \quad (7.12)$$

$$P_{13}(0,y,t)=\int\lambda_2(y)P_5(x,y,t)dx+\int2\lambda_4(y)P_{11}(x,y,t)dx \quad (7.13)$$

$$P_{14}(0,y,t)=\int\lambda_2(y)P_6(x,y,t)dx+\int\lambda_3(y)P_5(x,y,t)dx+\int2\lambda_4(y)P_{12}(x,y,t)dx \quad (7.14)$$

$$P_{15}(0,y,t)=\int\lambda_3(y)P_6(x,y,t)dx \quad (7.15)$$

On solving the differential Eqs.(5.1-5.16) together with initial conditions Eq.(6) and boundary conditions Eqs.(7.1-7.15), we can find all the probabilities and then the time dependent availability function $A(t)$ can be computed. For the validation of the results, the sum of the probabilities should be equal to one for each Markovian model, that is

$$P_0(t)+\sum_{i=1}^{15}P_i(x,y,t)=1 \quad (8)$$

The system of differential Eqs.(5.1-5.16) together with the initial conditions given by Eq.(6) and boundary conditions given by Eqs.(7.1-7.15) are called Chapman-Kolmogorov differential difference equations. Eq.(5.1) is a linear differential equation of first order and other Eqs. (5.2-5.16) are linear partial differential equations. In order to find the availability of the system, the governing Eqs.(5.1-5.16) along with the initial conditions given by Eq.(2) and boundary conditions given by Eqs.(7.1-7.15) can be solved using the Lagrange's method following the approach of [9] to get the state probabilities $P_0(t)$ and $P_i(x,y,t)$,$(i=1,2,\ldots,15)$.

$$P_0(t)=e^{-3\sum_{i=1}^{3}\lambda_i(y)t}\left[1+\int C_0(t)e^{3\sum_{i=1}^{3}\lambda_i(y)t}dt\right] \quad (9.1)$$

$$P_1(x,y,t)=e^{-\int T_1(x,y)dx}\left[\int C_1(x,y,t)e^{\int T_1(x,y)dx}dx+3\lambda_1(y-x)P_0(t-x)\right] \quad (9.2)$$

$$P_2(x,y,t)=e^{-\int T_2(x,y)dx}\left[\int C_2(x,y,t)e^{\int T_2(x,y)dx}dx+3\lambda_2(y-x)P_0(t-x)+\int\lambda_4(y-x)P_1(x,y-x,t-x)dx\right] \quad (9.3)$$

$$P_3(x,y,t)=e^{-\int T_3(x,y)dx}\left[\int C_3(x,y,t)e^{\int T_3(x,y)dx}dx+3\lambda_3(y-x)P_0(t-x)\right] \quad (9.4)$$

$$P_4(x,y,t)=e^{-\int T_4(x,y)dx}\left[\int C_4(x,y,t)e^{\int T_4(x,y)dx}dx+\int2\lambda_1(y-x)P_1(x,y-x,t-x)dx\right]$$
(9.5)

$$P_5(x,y,t)=e^{-\int T_5(x,y)dx}\left[\int C_5(x,y,t)e^{\int T_5(x,y)dx}dx+\int2\lambda_2(y-x)P_1(x,y-x,t-x)dx+\int2\lambda_4(y-x)P_4(x,y-x,t-x)dx+\int2\lambda_1(y-x)P_2(x,y-x,t-x)dx\right] \quad (9.6)$$

$$P_6(x,y,t)=e^{-\int T_6(x,y)dx}\left[\int C_6(x,y,t)e^{\int T_6(x,y)dx}dx+\int2\lambda_1(y-x)P_3(x,y-x,t-x)dx+\int2\lambda_3(y-x)P_1(x,y-x,t-x)dx\right] \quad (9.7)$$

$$P_7(x,y,t)=e^{-\int T_7(x,y)dx}\left[\int C_7(x,y,t)e^{\int T_7(x,y)dx}dx+\int2\lambda_2(y-x)P_2(x,y-x,t-x)dx+\int\lambda_4(y-x)P_5(x,y-x,t-x)dx\right] \quad (9.8)$$

$$P_8(x,y,t)=e^{-\int T_8(x,y)dx}\left[\int C_8(x,y,t)e^{\int T_8(x,y)dx}dx+\int2\lambda_2(y-x)P_3(x,y-x,t-x)dx+\int2\lambda_3(y-x)P_2(x,y-x,t-x)dx+\int\lambda_4(y-x)P_6(x,y-x,t-x)dx\right] \quad (9.9)$$

$$P_9(x,y,t)=e^{-\int T_9(x,y)dx}\left[\int C_9(x,y,t)e^{\int T_9(x,y)dx}dx+\int2\lambda_3(y-x)P_3(x,y-x,t-x)dx\right] \quad (9.10)$$

$$P_{10}(x,y,t)=e^{-\int T_{10}(x,y)dx}\Big[\int C_{10}(x,y,t)e^{\int T_{10}(x,y)dx}dx + \int \lambda_1(y-x)P_4(x,y-x,t-x)dx\Big] \quad (9.11)$$

$$P_{11}(x,y,t)=e^{-\int T_{11}(x,y)dx}\Big[\int C_{11}(x,y,t)e^{\int T_{11}(x,y)dx}dx + \int \lambda_1(y-x)P_5(x,y-x,t-x)dx + \int \lambda_2(y-x)P_4(x,y-x,t-x)dx + \int 3\lambda_4(y-x)P_{10}(x,y-x,t-x)dx\Big] \quad (9.12)$$

$$P_{12}(x,y,t)=e^{-\int T_{12}(x,y)dx}\Big[\int C_{12}(x,y,t)e^{\int T_{12}(x,y)dx}dx + \int \lambda_1(y-x)P_6(x,y-x,t-x)dx + \int \lambda_3(y-x)P_4(x,y-x,t-x)dx\Big] \quad (9.13)$$

$$P_{13}(x,y,t)=e^{-\int T_{13}(x,y)dx}\Big[\int C_{13}(x,y,t)e^{\int T_{13}(x,y)dx}dx + \int \lambda_2(y-x)P_5(x,y-x,t-x)dx + \int 2\lambda_4(y-x)P_{11}(x,y-x,t-x)dx\Big] \quad (9.14)$$

$$P_{14}(x,y,t)=e^{-\int T_{14}(x,y)dx}\Big[\int C_{14}(x,y,t)e^{\int T_{14}(x,y)dx}dx + + \int \lambda_2(y-x)P_6(x,y-x,t-x)dx + \int \lambda_3(y-x)P_5(x,y-x,t-x)dx + \int 2\lambda_4(y-x)P_{12}(x,y-x,t-x)dx\Big] \quad (9.15)$$

$$P_{15}(x,y,t)=e^{-\int T_{15}(x,y)dx}\Big[\int C_{15}(x,y,t)e^{\int T_{15}(x,y)dx}dx + \int \lambda_3(y-x)P_6(x,y-x,t-x)dx\Big] \quad (9.16)$$

From Eqs.(9.1-9.16), all the probabilities can be obtained. The time dependent availability $A(t)$ of the system is given as:

$$A(t)=P_0(t)+\int P_1(x,y,t)dx\,dy+\int P_2(x,y,t)dx\,dy + \int P_3(x,y,t)dx\,dy+\int P_4(x,y,t)dx\,dy + \int P_5(x,y,t)dx\,dy+\int P_6(x,y,t)dx\,dy + \int P_{10}(x,y,t)dx\,dy+\int P_{11}(x,y,t)dx\,dy + \int P_{12}(x,y,t)dx\,dy \quad (10)$$

As a special case we shall now discuss how to develop Chapman-Kolmogorov differential equations in transient as well as steady states when both failure and repair rates are constants.

4.2 Transient State When Both Failure And Repair Rates Are Constants

When both failure and repair rates are constants then $\frac{\partial}{\partial x},\frac{\partial}{\partial y}\to 0$. Consequently Eqs.(5.1-5.16) reduces to the ordinary linear differential equations which are given below:

$$\left(\frac{d}{dt}+3\sum_{i=1}^{3}\lambda_i\right)P_0(t)=\mu_1P_1(t)+\mu_2P_2(t)+\mu_3P_3(t) \quad (11.1)$$

$$\left(\frac{d}{dt}+2\sum_{i=1}^{3}\lambda_i+\lambda_4+\mu_1\right)P_1(t)=2\mu_1P_4(t)+\mu_2P_5(t)+\mu_3P_6(t)+3\lambda_1P_0(t) \quad (11.2)$$

$$\left(\frac{d}{dt}+2\sum_{i=1}^{3}\lambda_i+\mu_2\right)P_2(t)=\mu_1P_5(t)+2\mu_2P_7(t)+\mu_3P_8(t)+3\lambda_2P_0(t)+\lambda_4P_1(t) \quad (11.3)$$

$$\left(\frac{d}{dt}+2\sum_{i=1}^{3}\lambda_i+\mu_3\right)P_3(t)=\mu_1P_6(t)+\mu_2P_8(t)+2\mu_3P_9(t)+3\lambda_3P_0(t) \quad (11.4)$$

$$\left(\frac{d}{dt}+\sum_{i=1}^{3}\lambda_i+2\lambda_4+2\mu_1\right)P_4(t)=3\mu_1P_{10}(t)+\mu_2P_{11}(t)+\mu_3P_{12}(t)+2\lambda_1P_1(t) \quad (11.5)$$

$$\left(\frac{d}{dt}+\sum_{i=1}^{3}\lambda_i+\lambda_4+\mu_1+\mu_2\right)P_5(t)=2\mu_1P_{11}(t)+2\mu_2P_{13}(t)+\mu_3P_{14}(t)+2\lambda_2P_1(t)+2\lambda_4P_4(t)+2\lambda_1P_2(t) \quad (11.6)$$

$$\left(\frac{d}{dt}+\sum_{i=1}^{3}\lambda_i+\lambda_4+\mu_1+\mu_3\right)P_6(t)=2\mu_1P_{12}(t)+\mu_2P_{14}(t)+2\mu_3P_{15}(t)+2\lambda_1P_3(t)+2\lambda_3P_1(t) \quad (11.7)$$

$$\left(\frac{d}{dt}+2\mu_2\right)P_7(t)=2\lambda_2P_2(t)+\lambda_4P_5(t) \quad (11.8)$$

$$\left(\frac{d}{dt}+\mu_2+\mu_3\right)P_8(t)=2\lambda_2P_3(t)+2\lambda_3P_2(t)+\lambda_4P_6(t) \quad (11.9)$$

$$\left(\frac{d}{dt}+2\mu_3\right)P_9(t)=2\lambda_3P_3(t) \quad (11.10)$$

$$\left(\frac{d}{dt}+3\mu_1+3\lambda_4\right)P_{10}(t)=\lambda_1 P_4(t) \quad (11.11)$$

$$\left(\frac{d}{dt}+2\mu_1+\mu_2+2\lambda_4\right)P_{11}(t)=\lambda_1 P_5(t)+\lambda_2 P_4(t)+3\lambda_4 P_{10}(t) \quad (11.12)$$

$$\left(\frac{d}{dt}+2\mu_1+\mu_3+2\lambda_4\right)P_{12}(t)=\lambda_1 P_6(t)+\lambda_3 P_4(t) \quad (11.13)$$

$$\left(\frac{d}{dt}+2\mu_2\right)P_{13}(t)=\lambda_2 P_5(t)+2\lambda_4 P_{11}(t) \quad (11.14)$$

$$\left(\frac{d}{dt}+\mu_2+\mu_3\right)P_{14}(t)=\lambda_2 P_6(t)+\lambda_3 P_5(t)+2\lambda_4 P_{12}(t) \quad (11.15)$$

$$\left(\frac{d}{dt}+2\mu_3\right)P_{15}(t)=\lambda_3 P_6(t) \quad (11.16)$$

Initial conditions:

The initial conditions of the system are as follows:

$$P_i(0)=\begin{cases}1 & ,i=0\\ 0 & ,i=1,2,....,15\end{cases} \quad (12)$$

One can obtain the state probabilities by solving the system of differential Eqs.(11.1-11.16) together with the initial conditions Eq.(12) using Laplace transformation method. We obtain:

$$\left(s+3\sum_{i=1}^{3}\lambda_i\right)P_0(s)-\mu_1 P_1(s)-\mu_2 P_2(s)-\mu_3 P_3(s)=1 \quad (13.1)$$

$$\left(s+2\sum_{i=1}^{3}\lambda_i+\lambda_4+\mu_1\right)P_1(s)-2\mu_1 P_4(s)-\mu_2 P_5(s)-\mu_3 P_6(s)-3\lambda_1 P_0(s)=0 \quad (13.2)$$

$$\left(s+2\sum_{i=1}^{3}\lambda_i+\mu_2\right)P_2(s)-\mu_1 P_5(s)-2\mu_2 P_7(s)-\mu_3 P_8(s)-3\lambda_2 P_0(s)-\lambda_4 P_1(s)=0 \quad (13.3)$$

$$\left(s+2\sum_{i=1}^{3}\lambda_i+\mu_3\right)P_3(s)-\mu_1 P_6(s)-\mu_2 P_8(s)-2\mu_3 P_9(s)-3\lambda_3 P_0(s)=0 \quad (13.4)$$

$$\left(s+\sum_{i=1}^{3}\lambda_i+2\lambda_4+2\mu_1\right)P_4(s)-3\mu_1 P_{10}(s)-\mu_2 P_{11}(s)-\mu_3 P_{12}(s)-2\lambda_1 P_1(s)=0 \quad (13.5)$$

$$\left(s+\sum_{i=1}^{3}\lambda_i+\lambda_4+\mu_1+\mu_2\right)P_5(s)-2\mu_1 P_{11}(s)-2\mu_2 P_{13}(s)-\mu_3 P_{14}(s)-2\lambda_2 P_1(s)-2\lambda_4 P_4(s)-2\lambda_1 P_2(s)=0 \quad (13.6)$$

$$\left(s+\sum_{i=1}^{3}\lambda_i+\lambda_4+\mu_1+\mu_3\right)P_6(s)-2\mu_1 P_{12}(s)-\mu_2 P_{14}(s)-2\mu_3 P_{15}(s)-2\lambda_1 P_3(s)-2\lambda_3 P_1(s)=0 \quad (13.7)$$

$$(s+2\mu_2)P_7(s)-2\lambda_2 P_2(s)-\lambda_4 P_5(s)=0 \quad (13.8)$$

$$(s+\mu_2+\mu_3)P_8(s)-2\lambda_2 P_3(s)-2\lambda_3 P_2(s)-\lambda_4 P_6(s)=0 \quad (13.9)$$

$$(s+2\mu_3)P_9(s)-2\lambda_3 P_3(s)=0 \quad (13.10)$$

$$(s+3\mu_1+3\lambda_4)P_{10}(s)-\lambda_1 P_4(s)=0 \quad (13.11)$$

$$(s+2\mu_1+\mu_2+2\lambda_4)P_{11}(s)-\lambda_1 P_5(s)-\lambda_2 P_4(s)-3\lambda_4 P_{10}(s)=0 \quad (13.12)$$

$$(s+2\mu_1+\mu_3+2\lambda_4)P_{12}(s)-\lambda_1 P_6(s)-\lambda_3 P_4(s)=0 \quad (13.13)$$

$$(s+2\mu_2)P_{13}(s)-\lambda_2 P_5(s)-2\lambda_4 P_{11}(s)=0 \quad (13.14)$$

$$(s+\mu_2+\mu_3)P_{14}(s)-\lambda_2 P_6(s)-\lambda_3 P_5(s)-2\lambda_4 P_{12}(s)=0 \quad (13.15)$$

$$(s+2\mu_3)P_{15}(s)-\lambda_3 P_6(s)=0 \quad (13.16)$$

But it is difficult to find inverse Laplace transformation of Eqs.(13.1–13.16) since expressions for probability transforms are in very complicated form and the complexity increases with the increase in number of equations. To overcome such type of problems, the system of Eqs.(13.1–13.16) has been solved numerically to obtain the Laplace transformations $P_i(s)$, $i=0,1,2,....,15$ and the inverse Laplace transformation. The availability $A(t)$ of the system can be computed as:

$$A(t)=P_0(t)+P_1(t)+P_2(t)+P_3(t)+P_4(t)+P_5(t)+P_6(t)+P_{10}(t)+P_{11}(t)+P_{12}(t) \quad (14)$$

4.3 Steady State Availability When Both Failure And Repair Rates Are Constants

Management is always interested in steady state availability to achieve their optimal target. This can be obtained mathematically by taking $\frac{d}{dt}\to 0$ as $t\to\infty$ in the system of Eqs.(11.1-11.16) therefore, the system of Eqs.(11.1-11.16) reduces to the following system of linear equations:

$$\left(3\sum_{i=1}^{3}\lambda_i\right)P_0-\mu_1 P_1-\mu_2 P_2-\mu_3 P_3=0 \quad (15.1)$$

$$\left(2\sum_{i=1}^{3}\lambda_i+\lambda_4+\mu_1\right)P_1-2\mu_1 P_4-\mu_2 P_5-\mu_3 P_6-3\lambda_1 P_0=0 \quad (15.2)$$

$$\left(2\sum_{i=1}^{3}\lambda_i+\mu_2\right)P_2-\mu_1P_5-2\mu_2P_7-\mu_3P_8-3\lambda_2P_0-\lambda_4P_1=0 \quad (15.3)$$

$$\left(2\sum_{i=1}^{3}\lambda_i+\mu_3\right)P_3-\mu_1P_6-\mu_2P_8-2\mu_3P_9-3\lambda_3P_0=0 \quad (15.4)$$

$$\left(\sum_{i=1}^{3}\lambda_i+2\lambda_4+2\mu_1\right)P_4-3\mu_1P_{10}-\mu_2P_{11}-\mu_3P_{12}-2\lambda_1P_1=0 \quad (15.5)$$

$$\left(\sum_{i=1}^{3}\lambda_i+\lambda_4+\mu_1+\mu_2\right)P_5-2\mu_1P_{11}-2\mu_2P_{13}-\mu_3P_{14}-2\lambda_2P_1-2\lambda_4P_4-2\lambda_1P_2=0 \quad (15.6)$$

$$\left(\sum_{i=1}^{3}\lambda_i+\lambda_4+\mu_1+\mu_3\right)P_6-2\mu_1P_{12}-\mu_2P_{14}-2\mu_3P_{15}-2\lambda_1P_3-2\lambda_3P_1=0 \quad (15.7)$$

$$2\mu_2P_7-2\lambda_2P_2-\lambda_4P_5=0 \quad (15.8)$$

$$(\mu_2+\mu_3)P_8-2\lambda_2P_3-2\lambda_3P_2-\lambda_4P_6=0 \quad (15.9)$$

$$2\mu_3P_9-2\lambda_3P_3=0 \quad (15.10)$$

$$(3\mu_1+3\lambda_4)P_{10}-\lambda_1P_4=0 \quad (15.11)$$

$$(2\mu_1+\mu_2+2\lambda_4)P_{11}-\lambda_1P_5-\lambda_2P_4-3\lambda_4P_{10}=0 \quad (15.12)$$

$$(2\mu_1+\mu_3+2\lambda_4)P_{12}-\lambda_1P_6-\lambda_3P_4=0 \quad (15.13)$$

$$2\mu_2P_{13}-\lambda_2P_5-2\lambda_4P_{11}=0 \quad (15.14)$$

$$(\mu_2+\mu_3)P_{14}-\lambda_2P_6-\lambda_3P_5-2\lambda_4P_{12}=0 \quad (15.15)$$

$$2\mu_3P_{15}-\lambda_3P_6=0 \quad (15.16)$$

Solving the system of linear Eqs.(15.1-15.16) using Maple program, we get the state probabilities determining the steady state availability of the system in terms of P_{10}. These are obtained as follows:

$$P_0=\frac{\left(3\mu_1\lambda_4^2+3\mu_1^2\lambda_4+\mu_1^3+\lambda_4^3\right)}{\lambda_1^3}P_{10} \quad (16.1)$$

$$P_1=\frac{3}{\lambda_1^2}\left(\lambda_4^2+2\mu_1\lambda_4+\mu_1^2\right)P_{10} \quad (16.2)$$

$$P_2=\frac{3}{\mu_2\lambda_1^3}\left[2\lambda_4^2\lambda_1\mu_1+\lambda_4\lambda_1\mu_1^2+\lambda_4^3\lambda_1+3\mu_1\lambda_4^2\lambda_2+3\mu_1^2\lambda_4\lambda_2+\mu_1^3\lambda_2+\lambda_4^3\lambda_2\right]P_{10} \quad (16.3)$$

$$P_3=\frac{3\lambda_3}{\lambda_1^3\mu_3}\left(3\mu_1\lambda_4^2+3\mu_1^2\lambda_4+\mu_1^3+\lambda_4^3\right)P_{10} \quad (16.4)$$

$$P_4=\frac{3(\mu_1+\lambda_4)}{\lambda_1}P_{10} \quad (16.5)$$

$$P_5=\frac{6}{\mu_2\lambda_1^2}\left(\lambda_4^2\lambda_2+\lambda_4^2\lambda_1+2\lambda_4\lambda_2\mu_1+\lambda_4\lambda_1\mu_1+\lambda_2\mu_1^2\right)P_{10} \quad (16.6)$$

$$P_6=\frac{6\lambda_3}{\lambda_1^2\mu_3}\left(\lambda_4^2+2\mu_1\lambda_4+\mu_1^2\right)P_{10} \quad (16.7)$$

$$P_7=\frac{3}{\mu_2^2\lambda_1^3}\left[4\lambda_4^2\mu_1\lambda_2\lambda_1+2\lambda_4\mu_1^2\lambda_2\lambda_1+2\lambda_4^3\lambda_2\lambda_1+3\lambda_4^2\mu_1\lambda_2^2+3\lambda_4\mu_1^2\lambda_2^2+\mu_1^3\lambda_2^2+\lambda_4^3\lambda_2^2+\lambda_4^3\lambda_1^2+\lambda_4^2\lambda_1^2\mu_1\right]P_{10} \quad (16.8)$$

$$P_8=\frac{6\lambda_3}{\lambda_1^3\mu_3\mu_2}\left[2\lambda_4^2\lambda_1\mu_1+\lambda_4\lambda_1\mu_1^2+\lambda_4^3\lambda_1+3\mu_1\lambda_4^2\lambda_2+3\mu_1^2\lambda_4\lambda_2+\mu_1^3\lambda_2+\lambda_4^3\lambda_2\right]P_{10} \quad (16.9)$$

$$P_9=\frac{2\lambda_3^2}{\lambda_1^3\mu_3^2}\left(3\mu_1\lambda_4^2+3\mu_1^2\lambda_4+\mu_1^3+\lambda_4^3\right)P_{10} \quad (16.10)$$

$$P_{10}=P_{10} \quad (16.11)$$

$$P_{11}=\frac{3}{\lambda_1\mu_2}\left(\lambda_2\mu_1+\lambda_4\lambda_2+\lambda_1\lambda_4\right)P_{10} \quad (16.12)$$

$$P_{12}=\frac{3\lambda_3}{\lambda_1\mu_3}\left(\mu_1+\lambda_4\right)P_{10} \quad (16.13)$$

$$P_{13}=\frac{3}{\mu_2^2\lambda_1^2}\left[\lambda_4^2\lambda_2^2+2\lambda_4^2\lambda_1\lambda_2+2\mu_1\lambda_4\lambda_2^2+2\mu_1\lambda_4\lambda_1\lambda_2+\mu_1^2\lambda_2^2+\lambda_1^2\lambda_4^2\right]P_{10} \quad (16.14)$$

$$P_{14}=\frac{6\lambda_3}{\mu_2\lambda_1^2\mu_3}\left[\lambda_4^2\lambda_2+\lambda_4^2\lambda_1+2\lambda_4\lambda_2\mu_1+\lambda_4\lambda_1\mu_1+\lambda_2\mu_1^2\right]P_{10} \quad (16.15)$$

$$P_{15}=\frac{3\lambda_3^2}{\lambda_1^2\mu_3^2}\left(\lambda_4^2+2\mu_1\lambda_4+\mu_1^2\right)P_{10} \quad (16.16)$$

, and using the normalization condition $\sum_{i=0}^{15}P_i=1$ we can get P_{10}.

Thus, the steady state availability of the system is obtained as:

$$A(\infty)=P_0+P_1+P_2+P_3+P_4+P_5+P_6+P_{10}+P_{11}+P_{12} \quad (17)$$

5.3 NUMERICAL SOLUTIONS

For the above particular cases when both failure and repair rates are constants, the numerical results of the availability of the 2-out-of-3: G system and the steady state availability are given as follows:

5.1 Transient State When Both Failure And Repair Rates Are Variables

The numerically computations have been carried out starting from time $t = 0$ to $t = 50$.
The failure and repair rates of each component are given by:

$\mu_1 = 0.10$, $\mu_2 = 0.07$, $\mu_3 = 0.09$

$\lambda_1 = 0.06$, $\lambda_2 = 0.02$, $\lambda_3 = 0.04$, $\lambda_4 = 0.08$

Using Maple program, the 2-out-of-3: G system availability given by Eq.(14) versus time is shown in Fig .2.

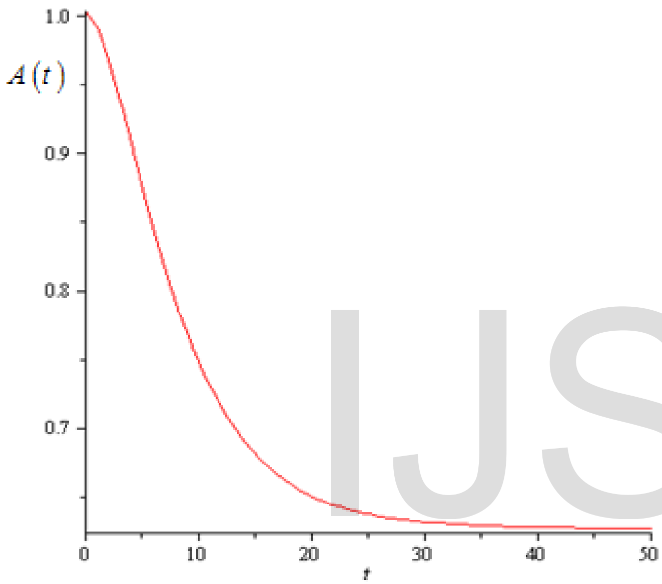

Fig. 2: system availability $A(t)$ versus time t

5.2 Steady State Availability When Both Failure And Repair Rates Are Constants

The system of linear Eqs.(15.1-15.15) has been solved numerically using Maple program using the above values of the failure and repair rates of each component.

The obtained values of the steady state probabilities are as follows:

$P_0 = 0.0756$,$P_1 = 0.0756$,$P_2 = 0.1511$,$P_3 = 0.1008$
$P_4 = 0.0252$,$P_5 = 0.1008$,$P_6 = 0.0672$,$P_7 = 0.1008$
$P_8 = 0.1344$,$P_9 = 0.0448$,$P_{10} = 0.0028$,$P_{11} = 0.0168$
$P_{12} = 0.0112$,$P_{13} = 0.0336$,$P_{14} = 0.0448$,$P_{15} = 0.015$

Thus the steady state availability as defined in Equation Eq.(17) is given by:

$$A(\infty) = P_0 + P_1 + P_2 + P_3 + P_4 + P_5 + P_6 + P_{10} + P_{11} + P_{12}$$
$$= 0.6268$$

6 CONCLUSION

The main objective of this paper was to offer a methodology for analyzing the availability for the k-out-of-n: G system with three types of failure rates; partial failure, complete failure under repair and complete failure under maintenance, using Markov model and supplementary variable technique. 2-out-of-3:G system is presented in order to illustrate the performance of the model. The problem of evaluating the availability of the system was formulated in a set of first order partial differential equations form, which seems convenient for computation with software packages like Maple program. Lagrange's method is used in this model to evaluate the state probabilities from the set of first order partial differential equations along with the initial conditions and boundary conditions when the failure and repair rates are variables. When both failure and repair rates are constants, the system of differential equations with the initial conditions has been solved using Laplace transformation using Maple program. Finally, the long run availability, steady state availability when both failure and repair rates are constants, has been computed for given values of failure and repair rates.

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